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Induction of quantum group representations

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Abstract

In this paper we will define a generalized procedure of induction of quantum group representations both from quantum and from coisotropic subgroups proving also their main properties. We will then show that such a procedure realizes quantum group representations on generalized quantum bundles.

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1. Introduction

Among the various methods to construct representations of a given Lie group, the induction procedure plays with no doubt a central role. On the one hand it gives a strong insight into the representation theory of a large class of non-semisimple Lie groups. For example, it solves completely the semidirect product case, linked as it is to the orbit method for nilpotent and solvable groups. On the other hand it gives foundations to the geometric theory of representations of semisimple Lie groups, from Borel–Weil–Bott theorem (see for example [19]). Induction of quantum group representations was developed in [16] and, later, in [13]. Recently a very detailed work, [14], summarizes known results for the compact case and in [2] some results on infinite-dimensional induced representations for non-semisimple quantum groups are obtained. The quantum theory, up to now, gives much less insight than its classical counterpart and suffers from a strong limitation: the extreme rarity of quantum subgroups [17]. For example [15,18] deal with induced representations

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for some non-semisimple quantum groups without approaching and solving the problem that they do not fit well the Parshall–Wang setting because they do not start with a well-defined quantum subgroup. To overcome this problem the theory of coisotropic quantum subgroups introduced in [9] seems useful and unavoidable.

Let us briefly recall some basics of the theory of subgroups of quantum groups, as developed in [4,9] (to which we refer for further motivations). A more detailed account of the problems we will deal with can also be found in [10], while for the general theory of quantum embeddable homogeneous spaces we refer to [11].

Definition 1.1. Given a Hopf algebra A_q we will call quantum subgroup of A_q any pair (B_q, π) such that B_q is a Hopf algebra and $\pi : A_q \twoheadrightarrow B_q$ is a Hopf algebra epimorphism. We will call coisotropic quantum left (resp. right) subgroup any pair (C_q, σ) such that C_q is a coalgebra and a left (resp. right) A_q-module and $\sigma : A_q \twoheadrightarrow C_q$ is a surjective linear map which is also a coalgebra and a left (resp. right) A_q-module homomorphism.

As usual in the classical case subgroups can be identified to the kernel of the restriction epimorphism. With this identification quantum subgroups are in 1:1 correspondence with Hopf ideals in A_q while left (right) coisotropic quantum subgroups are in 1:1 correspondence with bilateral coideals which are also left (right) ideals.

The weaker hypothesis defining coisotropic quantum subgroups is not descending from a need of generality for its own sake. It is rather quite naturally imposed by a detailed analysis of the semiclassical limit, i.e. the underlying Poisson–Lie theory. In this limit quantum subgroups correspond to Poisson–Lie subgroups, a fairly rare object. The class of Poisson–Lie subgroups is not closed by conjugation: conjugating a Poisson–Lie group gives a coisotropic subgroup. Quite naturally, then, such subgroups play a role in the theory of Poisson homogeneous spaces. More precisely, every Poisson homogenous space with at least one zero-dimensional symplectic leaf is a quotient by a coisotropic subgroup. Furthermore, it has been proven by Etingof and Kazhdan [12] that any such quotient can be suitably quantized. Quantum coisotropic subgroups are in "good" correspondence with quantum embeddable homogeneous spaces (where good means that such correspondence is bijective provided some technical conditions are satisfied) and their semiclassical limit, as expected, gives exactly coisotropic subgroups. For these reasons they seem a good class of quantum subgroups to start with.

The link with embeddable quantum homogeneous spaces is given as follows:

Definition 1.2. Let A_q be a Hopf algebra. A right (resp. left) quantum embeddable homogeneous space of A_q is a subalgebra which is also a right (resp. left) coideal.

Proposition 1.3. *Let (C, σ) be a right (resp. left) coisotropic quantum subgroup of A_q. Then*

$$B_C = \{f \in A_q | (\sigma \otimes id)\Delta f = \sigma(1) \otimes f\}$$

is a quantum right embeddable homogeneous space and, respectively

$$B^C = \{f \in A_q | (id \otimes \sigma)\Delta f = f \otimes \sigma(1)\}$$

is a quantum left embeddable homogeneous space. Conversely let B_q be a right (resp. left) embeddable quantum homogeneous space; then the right (resp. left) ideal generated by $\{b-\varepsilon(b)1\}$ is a bilateral coideal in A_q and identifies a right (resp. left) coisotropic subgroup.

In this paper we generalize induced quantum group representations to coisotropic quantum subgroups. We also show how the corresponding corepresentation space of the function algebra can be intrepreted as the space of sections of an associated vector bundle to a principal coalgebra quantum bundle, this last concept being recently introduced by Brzeziński [5,8], generalizing the classical relation between induced representations and homogeneous vector bundles [19]. We will not deal with the measure-theoretic concepts needed to develop correctly all unitariness aspects, which are the subject of a forthcoming paper.

2. Quantum induction

In this section we define induced corepresentations from coisotropic quantum subgroups and prove their main properties. Let us begin recalling some definitions. Let A be a Hopf algebra and let V be a vector space (we are considering a fixed base field of zero characteristic). V is said to be a right corepresentation of A if there exists a linear map $\rho_{\mathrm{R}} : V \to V \otimes A$ such that

$$(id \otimes \varepsilon) \circ \rho_{\mathrm{R}} = id_V, \tag{2.1}$$

$$(id \otimes \Delta) \circ \rho_{\mathrm{R}} = (\rho_{\mathrm{R}} \otimes \mathrm{id}_A) \circ \rho_{\mathrm{R}}. \tag{2.2}$$

We will say that V is a left corepresentation of A if there exists a linear map $\rho_{\mathrm{L}} : V \to A \otimes V$ such that

$$(\varepsilon \otimes id) \circ \rho_{\mathrm{L}} = id_V, \tag{2.3}$$

$$(\Delta \otimes id) \circ \rho_{\mathrm{L}} = (id_A \otimes \rho_{\mathrm{L}}) \circ \rho_{\mathrm{L}}. \tag{2.4}$$

Let us remark explicitly that above definitions are well posed for any coalgebra A.

Let us consider now a fixed quantum group $A_q = A_q(G)$ and a coisotropic quantum subgroup $(C_q, \pi) = (A_q(K), \pi)$ (when we do not specify coisotropic subgroups to be left or right we mean the result is valid in both cases).

Proposition 2.1. *The map $R = (id \otimes \pi) \circ \Delta_G : A_q(G) \to A_q(G) \otimes A_q(K)$ defines a right corepresentation of $A_q(K)$ on $A_q(G)$. Similarly the map $L = (\pi \otimes id) \circ \Delta_G$ defines a left $A_q(K)$-corepresentation.*

Proof. We will directly verify conditions (2.1) and (2.2) for R. The proof for L proceeds along the same lines. Let us start with (2.2). Left-hand side equals

$$(id \otimes \Delta_K)(Rf) = \sum_{(f)} f_{(1)} \otimes (\Delta_K \circ \pi)(f_{(2)}) = \sum_{(f)} f_{(1)} \otimes \pi(f_{(2)}) \otimes \pi(f_{(3)}),$$

where we have used coassociativity of Δ_G together with the fact that π is a coalgebra morphism. As for the right-hand side

$$\begin{aligned}((R \otimes id) \circ R) f &= \sum_{(f)} R(f_{(1)}) \otimes \pi(f_{(2)}) = \sum_{(f)} (id \otimes \pi)(f_{(1)_{(1)}} \otimes f_{(1)_{(2)}}) \otimes \pi(f_{(2)}) \\ &= \sum_{(f)} f_{(1)} \otimes \pi(f_{(2)}) \otimes \pi(f_{(3)}),\end{aligned}$$

again from coassociativity; this proves (2.2). For what concerns (2.1) we have

$$\begin{aligned}(id \otimes \varepsilon_K)(Rf) &= \sum_{(f)} f_{(1)} \otimes (\varepsilon_K \circ \pi)(f_{(2)}) \\ &= \sum_{(f)} f_{(1)} \otimes \varepsilon_G(f_{(2)}) = ((id \otimes \varepsilon_G) \circ \Delta) f = f,\end{aligned}$$

which completes the proof. □

Lemma 2.2. *The following identities hold:*

$$(\Delta_G \otimes id) \circ R = (id \otimes R) \circ \Delta_G, \tag{2.5}$$

$$(id \otimes \Delta_G) \circ L = (L \otimes id) \circ \Delta_G. \tag{2.6}$$

Proof. The first identity follows from the chain of equalities:

$$\begin{aligned}(\Delta_G \otimes id)(Rf) &= \sum_{(f)} \Delta_G(f_{(1)}) \otimes \pi(f_{(2)}) = \sum_{(f)} f_{(1)} \otimes f_{(2)} \otimes \pi(f_{(3)}) \\ &= (id \otimes R)(\Delta_G f) = \sum_{(f)} f_{(1)} \otimes R(f_{(2)}) = \sum_{(f)} f_{(1)} \otimes f_{(2)} \otimes \pi(f_{(3)}).\end{aligned}$$

The second one can be proven in a similar way. □

Straightforward calculations allow to prove also the following:

Lemma 2.3. *Let $(A_q(K), \pi)$ be a left coisotropic subgroup; then the following multiplication properties hold true:*

$$L(fg) = \Delta_G(f) L(g), \quad R(fg) = \Delta f \cdot R(g) \qquad \forall f, g \in A_q(G), \tag{2.7}$$

where $\cdot$ denotes the action of $A_q(G) \otimes A_q(G)$ on $A_q(G) \otimes A_q(K)$ given by $(f \otimes g) \cdot (h \otimes c) = fh \otimes g \cdot c$. Similarly if $(A_q(K), \pi)$ is a right coisotropic subgroup we have:

$$L(fg) = L(f) \Delta_G(g), \quad R(fg) = R(f) \Delta_G(g) \qquad \forall f, g \in A_q(G). \tag{2.8}$$

Let us now start from a right corepresentation ρ_{R} and a left corepresentation ρ_{L} of the coalgebra $A_q(K)$ on the vector space V. Define

$$\mathrm{ind}_K^G(\rho_L) = \{F \in A_q(G) \otimes V | (R \otimes id)(F) = (id \otimes \rho_L)(F)\}, \tag{2.9}$$

$$\mathrm{ind}_K^G(\rho_R) = \{F \in V \otimes A_q(G) | (id \otimes L)F = (\rho_R \otimes id)(F)\}. \tag{2.10}$$

Both these spaces are kernels of certain linear operators and, as such, are closed vector subspaces of $A_q(G) \otimes V$ and $V \otimes A_q(G)$.

Proposition 2.4. *$\Delta_G \otimes id$ defines a left corepresentation of $A_q(G)$ on $\mathrm{ind}_K^G(\rho_L)$ and $id \otimes \Delta_G$ defines a right corepresentation of the same space on $\mathrm{ind}_K^G(\rho_R)$.*

Proof. Let us prove the first claim. To begin with we need to prove that $(\Delta_G \otimes id)(F) \in A_q(G) \otimes \mathrm{ind}_K^G(\rho_L)$. Linearity of all the operations involved implies that we can consider only vectors of the form $F = f \otimes v$. Then we need to prove that

$$(id \otimes R \otimes id)(\Delta_G \otimes id)(f \otimes v) = (id \otimes id \otimes \rho_L)(\Delta_G \otimes id)(f \otimes v).$$

Using the first formula in Lemma 2.2 the left-hand side equals

$$\begin{aligned}(((id \otimes R) \circ \Delta_G) \otimes id)(f \otimes v) &= (((\Delta_G \otimes id) \circ R) \otimes id)(f \otimes v) \\ &= (\Delta_G \otimes id \otimes id)(R \otimes id)(f \otimes v),\end{aligned}$$

which, recalling that $f \otimes v$ belongs to $\mathrm{ind}_K^G(\rho_L)$, equals

$$(\Delta_G \otimes id \otimes id)(id \otimes \rho_L)(f \otimes v) = (id \otimes id \otimes \rho_L)(\Delta_G \otimes id)(f \otimes v).$$

Now, the corepresentation condition (2.2) is equivalent to coassociativity of Δ_G and the condition (2.1) is equivalent to properties of the counity ε_G. The second claim follows similarly from the second formula in Lemma 2.2. □

Definition 2.5. Given a right (resp. left) corepresentation ρ_R (resp. ρ_L) of the quantum coisotropic subgroup $(A_q(K), \pi)$ the corresponding corepresentation $id \otimes \Delta_G$ on $\mathrm{ind}_K^G(\rho_R)$ (resp. $\Delta_G \otimes id$) on $\mathrm{ind}_K^G(\rho_L)$ of $A_q(G)$ is called induced representation from ρ_R (resp. ρ_L) on $A_q(G)$.

Remark. *In case ρ_L is a one-dimensional corepresentation the induced representation is given by*

$$\mathrm{ind}_K^G(\rho_L) = \{f \in A_q(G) | (R \otimes id)f = (id \otimes \rho_L)f\}$$

with coaction Δ_G. Similarly if ρ_R is one-dimensional

$$\mathrm{ind}_K^G(\rho_R) = \{f \in A_q(G) | (id \otimes L)f = (\rho_R \otimes id)f\}.$$

In both cases the induced representations can be seen as subrepresentations of the regular (left or right) representation $(A_q(G), \Delta_G)$. In agreement with the classical case we will call monomial such representations.

As usual in representation theory we will often identify the representation space with the representation itself, being clear which is the representation map.

Proposition 2.6. *If ρ_R and ρ'_R are equivalent right (resp. left) $A_q(K)$-corepresentations then $\mathrm{ind}_K^G(\rho_R)$ and $\mathrm{ind}_K^G(\rho'_R)$ are equivalent right (resp. left) $A_q(G)$-corepresentations.*

Proof. Let $\rho_R : V \to A_q(K) \otimes V$ and $\rho'_R : W \to A_q(K) \otimes W$ be equivalent. Then there exists a vector space isomorphism $F : V \to W$ such that $\rho'_R \circ F = (id \otimes F) \circ \rho_R$. Let us consider the vector space isomorphism

$$\tilde{F} = F \otimes id : V \otimes A_q(G) \to W \otimes A_q(G).$$

First note that $\tilde{F}(\mathrm{ind}_K^G(\rho_R)) \subset \mathrm{ind}_K^G(\rho'_R)$. Let $v \otimes f \in \mathrm{ind}_K^G(\rho_R)$. We want to prove that $F(v) \otimes f$ verifies

$$F(v) \otimes L(f) = \rho'_R(F(v)) \otimes f,$$

which follows from

$$\begin{aligned}\rho'_R(F(v)) \otimes f &= ((id \otimes F) \circ \rho_R)(v) \otimes f = (id \otimes F \otimes id)(\rho_R(v) \otimes f) \\ &= (F \otimes id \otimes id)(v \otimes L(f)) = F(v) \otimes L(f).\end{aligned}$$

Next we prove that the vector space isomorphism $\tilde{F}$ intertwines corepresentations, i.e.

$$(id \otimes \Delta_G) \circ \tilde{F} = (\tilde{F} \otimes id)(id \otimes \Delta_G),$$

which follows from

$$\begin{aligned}(id \otimes \Delta_G) \circ \tilde{F}(v \otimes f) &= (id \otimes \Delta_G)(F(v) \otimes f) = F(v) \otimes \Delta_G f \\ &= (\tilde{F} \otimes id)(v \otimes \Delta_G f) \\ &= (\tilde{F} \otimes id)(id \otimes \Delta_G)(v \otimes f). \qquad \square\end{aligned}$$

Another relevant property is the behaviour of the induction procedure with respect to direct sums. Let ρ_1 and ρ_2 be right corepresentations of the coalgebra C, respectively, on vector spaces V and W; we may then define the following right C-corepresentation:

$$\begin{aligned}&\rho_1 \oplus \rho_2 : V \oplus W \to C \otimes (V \oplus W) \\ &\rho_1 \oplus \rho_2 = (id \otimes \iota_V) \circ \rho_1 \circ p_V + (id \otimes \iota_W) \circ \rho_2 \circ p_W,\end{aligned} \tag{2.11}$$

where $\iota_V : V \to V \bigoplus W$ and $\iota_W : W \to V \bigoplus W$ are the natural immersion maps and $p_V : V \bigoplus W \to V$ and $p_W : V \bigoplus W \to W$ are the natural projections (with obvious modifications for direct sums of left corepresentations).

Proposition 2.7. *Let (ρ_k, V_k), $k \in \mathbb{N}$ be right (resp. left) $A_q(K)$-corepresentations of a given coisotropic quantum subgroup $(A_q(K), \pi)$ of $A_q(G)$. Then we have an equivalence of right (resp. left) corepresentations*

$$\mathrm{ind}_K^G\left(\bigoplus_k \rho_k\right) \equiv \bigoplus_k \mathrm{ind}_K^G(\rho_k).$$

In particular, if $\mathrm{ind}_K^G(\rho)$ is irreducible then ρ is irreducible.

Proof. Due to associativity of the direct sum it is sufficient to prove the theorem for $k = 1, 2$. We will begin proving that $p_V \otimes id + p_W \otimes id$ is an isomorphism of vector spaces of $\mathrm{ind}_K^G(\rho_1 \oplus \rho_2)$ in $\mathrm{ind}_K^G(\rho_1) \bigoplus \mathrm{ind}_K^G(\rho_2)$. Due to the fact that the above map is a linear isomorphism of $(V \bigoplus W) \otimes A_q(G)$ in $(V \otimes A_q(G)) \bigoplus (W \otimes A_q(G))$ it is sufficient to prove that $(p_V \otimes id)F \in \mathrm{ind}_K^G(\rho_1)$ and $(p_W \otimes id)F \in \mathrm{ind}_K^G(\rho_2)$. We have:

$$
\begin{aligned}
&(id \otimes L)(p_V \otimes id)F \\
&\quad = (p_V \otimes id \otimes id)(id \otimes L)F \\
&\quad = (p_V \otimes id \otimes id)((\rho_1 \oplus \rho_2) \otimes id)F = ((p_V \otimes id) \circ (\rho_1 \oplus \rho_2)) \otimes id(F) \\
&\quad = (p_V \otimes id)[(\iota_V \otimes id) \circ \rho_1 \circ p_V + (\iota_W \otimes id) \circ \rho_2 \circ p_W] \otimes id(F) \\
&\quad = [(\iota_V \otimes id) \circ \rho_1 \circ p_V \otimes id](F) = (\rho_1 \otimes id)(p_V \otimes id)F,
\end{aligned}
$$

and similarly for $(p_W \otimes id)$. We can then identify the vector spaces writing for every $f \in \mathrm{ind}_K^G(\rho_1 \oplus \rho_2)$, $f = f_1 + f_2$ with $f_1 \in \mathrm{ind}_K^G(\rho_1)$ and $f_2 \in \mathrm{ind}_K^G(\rho_2)$.

The intertwining property follows trivially from

$$(id \otimes \Delta_G)f = (id \otimes \Delta_G)(f_1 + f_2) = (id \otimes \Delta_G)f_1 + (id \otimes \Delta_G)f_2. \qquad \square$$

One more property of the classical induction procedure that we want to mimic is double induction, i.e. what happens when we start with two subgroups, one of which contains the other, and a representation of the smaller one. At this purpose let us remark that a coisotropic quantum subgroup cannot have a quantum subgroup but only coisotropic quantum subgroups. Let us consider then the case in which $A_q(K)$ is a quantum subgroup of $A_q(G)$, coisotropic or not, and $A_q(H)$ is a coisotropic quantum subgroup of $A_q(K)$.

Proposition 2.8. *Let ρ_{R} be a right (resp. left) $A_q(H)$-corepresentation on V. Then there is an equivalence of right (resp. left) $A_q(G)$-corepresentations between*

$$\mathrm{ind}_H^G(\rho_{\mathrm{R}}) \equiv \mathrm{ind}_K^G(\mathrm{ind}_H^K(\rho_{\mathrm{R}})).$$

Proof. Let us denote with π_{KH} and π_{GK}, respectively, the maps defining H as a quantum subgroup of K and K as a quantum subgroup of G and let $\pi_{GH} = \pi_{KH} \circ \pi_{GH}$ defining H as a quantum subgroup of G. Let us denote with L_{GK}, L_{KH} and L_{GH} the corresponding left corepresentations granted by Proposition 2.1. Let us remark that $L_{GH} = (\pi_{KH} \otimes id) \circ L_{GK}$. The required isomorphism between representation spaces is given by

$$id \otimes L_{GK} : \mathrm{ind}_H^G(\rho) \to \mathrm{ind}_K^G(\mathrm{ind}_H^K(\rho)).$$

To prove it we remark that elements of the second space are those $v \otimes g \otimes f$ in $V \otimes A_q(K) \otimes A_q(G)$ verifying

$$(1) \qquad \rho_{\mathrm{R}}(v) \otimes g \otimes f = \sum_{(g)} v \otimes \pi_{KH}(g_{(1)}) \otimes g_{(2)} \otimes f,$$

$$(2) \qquad v \otimes \Delta_K(g) \otimes f = \sum_{(f)} v \otimes g \otimes \pi_{GK}(f_{(1)}) \otimes f_{(2)}.$$

Applying $id \otimes \varepsilon_K \otimes id \otimes id$ to this second identity gives

$$v \otimes g \otimes f = \sum_{(f)} \varepsilon_K(g) v \otimes \pi_{GK}(f_{(1)}) \otimes f_{(2)},$$

from which one can prove that $v \otimes g \otimes f \mapsto \varepsilon(g) v \otimes f$ is both a left and a right inverse of $id \otimes L_{GK}$. The fact that the actions are intertwined results from coassociativity of Δ_G.

This proves the proposition. □

Let us lastly show how the induction procedure interacts with automorphisms of the Hopf algebra structure. Let $A_q(G)$ be the quantum group and $(A_q(K), \pi)$ a coisotropic quantum subgroup. Let $\alpha \in \mathrm{Aut}(A_q(G))$ be a Hopf-algebra automorphism. Then there exists one and only one coalgebra automorphism $\tilde{\alpha} : A_q(K) \to A_q(K)$ such that: $\tilde{\alpha} \circ \pi = \pi \circ \alpha$.

Proposition 2.9. *In the above hypothesis there exists an equivalence of $A_q(G)$-right corepresentations*

$$\mathrm{ind}_K^G((id \otimes \tilde{\alpha}) \circ \rho_{\mathrm{R}}) = (id \otimes \alpha)(\mathrm{ind}_K^G(\rho_{\mathrm{R}})).$$

Proof. The vector space of the left-hand side $A_q(G)$-corepresentation is

$$\{F \in V \otimes A_q(G) | (id \otimes L)F = (id \otimes \tilde{\alpha} \otimes id)(\rho_{\mathrm{R}} \otimes id)F\}.$$

If F belongs to $\mathrm{ind}_K^G(\rho)$ then

$$\begin{aligned}
&(id \otimes L)(id \otimes \alpha)F \\
&\quad = v \otimes L(\alpha(f)) = v \otimes ((\pi \otimes id)\Delta(\alpha(f))) \\
&\quad = v \otimes ((\pi \otimes id)(\alpha \otimes \alpha(\Delta f))) = v \otimes ((\tilde{\alpha} \otimes \tilde{\alpha})(\pi \otimes id(\Delta f))) \\
&\quad = (id \otimes \tilde{\alpha} \otimes \tilde{\alpha})(v \otimes L(f)) = (id \otimes \tilde{\alpha} \otimes \tilde{\alpha})(v \otimes \rho(f)) \\
&\quad = (id \otimes \tilde{\alpha})(\rho(v)) \otimes \alpha(f),
\end{aligned}$$

and this proves the isomorphism between the vector spaces carrying the representation. The intertwining property is a simple consequence of α being a coalgebra morphism. □

3. Geometric realization on homogeneous quantum bundles

The purpose of this section is to explicit the relations between induced corepresentations from coisotropic quantum subgroups and embeddable quantum homogeneous spaces. We will split the two cases of quantum and coisotropic subgroup, although the first one is a special case of the second, to point out the differences. Much of what follows is strongly related to results in Refs. [5,7,8], where the theory of quantum principal bundles with structure group given by a coalgebra has been developed. Those results are reinterpretated (and slightly generalized) here in the context of induced corepresentations. This reflects what happens in the classical case where there is a bijective correspondence between induced representations and homogeneous vector bundles (i.e. vector bundles associated to principal bundles [19]). For some additional details on the quantum bundle interpretation we refer to [14], although limited to the compact and less general quantum subgroup case.

Let $A_q(G)$ be a Hopf algebra with invertible antipode and let $(A_q(K), \pi)$ be a quantum subgroup. Let B_K be the corresponding quantum quotient space

$$B_K = \{f \in A_q(G) | (\pi \otimes id)\Delta(f) = 1 \otimes f\}.$$

Let ρ_{R} be a $A_q(K)$ right corepresentation on V and let $\mathrm{ind}_K^G(\rho_{\mathrm{R}})$ be the space of the induced $A_q(G)$-corepresentation.

Lemma 3.1. $\mathrm{ind}_K^G(\rho_{\mathrm{R}})$ *is a left and right* B_K*-module, with the action given by linear extension of*

$$b \cdot (f \otimes v) = (bf \otimes v), \qquad (f \otimes v) \cdot b = fb \otimes v.$$

Proof. Let $v \otimes f \in \mathrm{ind}_K^G(\rho_{\mathrm{R}})$. Then

$$\begin{aligned} L(v \otimes bf) &= \sum_{(b)(f)} v \otimes \pi(b_{(1)} f_{(1)}) \otimes b_{(2)} f_{(2)} \\ &= \sum_{(b)(f)} v \otimes \varepsilon(b_{(1)}) \pi(f_{(1)}) \otimes b_{(2)} \pi(f_{(2)}) \\ &= \sum_{(f)} v \otimes f_{(1)} \otimes bf_{(2)} = \sum_{(v)} v_{(0)} \otimes v_{(1)} \otimes bf = b \cdot (\rho_{\mathrm{R}} \otimes id)(v \otimes f). \end{aligned}$$

The same proof holds for the right action (π is an algebra morphism). □

Let us now recall that linear maps $f, g : A_q(K) \to A_q(G)$ can be multiplied according to convolution

$$(f * g)(k) = \sum_{(k)} f(k_{(1)}) g(k_{(2)}).$$

If f is a map from $A_q(K)$ to $A_q(G)$ its convolution inverse, if it exists, is a map f^{-1} : $A_q(K) \to A_q(G)$ such that

$$\sum_{(k)} f(k_{(1)}) f^{-1}(k_{(2)}) = \varepsilon(k) 1 = \sum_{(k)} f^{-1}(k_{(1)}) f(k_{(2)}).$$

Definition 3.2. A quantum subgroup is said to have a left (resp. right) section if there exists a linear, convolution invertible, map $\phi : A_q(K) \to A_q(G)$ such that:

(i) $\phi(1) = 1$;

(ii) $(\pi \otimes id)\Delta_G \circ \phi = (1 \otimes \phi) \circ \Delta_K$ (or, respectively (ii′)) $(id \otimes \pi)\Delta_G \circ \phi = (\phi \otimes 1) \circ \Delta_K$.

If, furthermore, ϕ is an algebra morphism then $(A_q(K), \pi)$ is said to be trivializable.

The second condition is an intertwining condition between the corepresentation of $A_q(K)$ on itself and its corepresentation on $A_q(G)$.

Proposition 3.3. *Let us suppose that* $(A_q(K), \pi)$ *has a section* ϕ. *Then:*
(i) $A_q(G)$ *is isomorphic to* $A_q(K) \otimes B_K$ *as a vector space;*
(ii) $\mathrm{ind}_K^G(\rho_R)$ *is isomorphic to* $V \otimes B_K$ *as a* B_K*-module (both left and right).*

Proof. First of all let us consider the following lemma, which can be proved with usual Hopf algebra techniques.

Lemma 3.4. *The convolution inverse of the section* ϕ *verifies*

$$(\pi \otimes id)\Delta_G \phi^{-1} = (S \otimes \phi^{-1})\tau_{1,2}\Delta_K,$$

where $\tau_{1,2}$ *is the map interchanging terms in tensor product.*

Let us now start from $f \in A_q(G)$. Then

$$f = \sum_{(f)} \varepsilon(f_{(1)}) f_{(2)} = \sum_{(f)} \varepsilon(\pi(f_{(1)})) f_{(2)} = \sum_{(f)} \phi(\pi(f_{(1)}))\phi^{-1}(\pi(f_{(2)})) f_{(3)}.$$

Let us now prove that $\sum_{(f)} \phi^{-1}(\pi(f_{(1)})) f_{(2)}$ belongs to B_K.

$$\begin{aligned}
&(\pi \otimes id)\Delta\left(\sum_{(f)} \phi^{-1}(\pi(f_{(1)})) f_{(2)}\right) \\
&= \sum_{(f)} \pi(\phi^{-1}(\pi(f_{(1)}))_{(1)}) f_{(2)_{(1)}} \otimes (\phi^{-1}(\pi(f_{(1)}))_{(2)} f_{(2)})_{(2)} \\
&= \sum_{(f)} S_K(\pi(f_{(2)}))\pi(f_{(3)}) \otimes \phi^{-1}(\pi(f_{(1)})) f_{(4)} \\
&= \sum_{(f)} 1 \otimes \varepsilon(f_{(2)})\phi^{-1}(\pi(f_{(1)})) f_{(3)} \\
&= 1 \otimes \sum_{(f)} \phi^{-1}(\pi(f_{(1)})) f_{(2)}.
\end{aligned}$$

This proves that the map $A_\phi : A_q(K) \otimes B_K \to A_q(G)$ linearly extending $k \otimes b \mapsto \phi(k)b$ is surjective and its right inverse is given by

$$A_\phi^{-1} : f \mapsto \sum_{(f)} \pi(f_{(1)}) \otimes \phi^{-1}(\pi(f_{(2)})) f_{(3)}.$$

Let us verify that A_ϕ^{-1} is also a left inverse: in fact

$$A_\phi^{-1}(\phi(k)b) = \sum_{(b)(\phi(k))} \pi(\phi(k)_{(1)} b_{(1)}) \otimes \phi^{-1}(\pi(\phi(k)_{(2)} b_{(2)}))\phi(k)_{(3)} b_{(3)}$$

using the fact that (ii) of Definition 3.2 implies that $(\pi \otimes \pi \otimes id)(\Delta_G \otimes id)\Delta_G \phi$ equals $(id \otimes id \otimes \phi)(\Delta_K \otimes id)\Delta_K$ we have

$$\sum_{(\phi(k))} \pi(\phi(k)_{(1)}) \otimes \phi^{-1}(\pi(\phi(k)_{(2)}))\phi(k)_{(3)}b = \sum_{(k)} k_{(1)} \otimes \phi^{-1}(k_{(2)})\phi(k_{(3)})b$$
$$= \sum_{(k)} \varepsilon(k_{(2)})k_{(1)} \otimes b = k \otimes b,$$

which proves the claim.

The second isomorphism, although more involved, can still be verified explicitly defining it and its inverse. Let us first consider

$$T_\phi : V \to \mathrm{ind}_K^G(\rho_\mathrm{R}), \qquad v \mapsto \sum_{(v)} v_{(0)} \otimes \phi(v_{(1)}).$$

We want T_ϕ to take values in $\mathrm{ind}_K^G(\rho_\mathrm{R})$. This is true if and only if

$$\sum_{(v)} \rho_\mathrm{R}(v_{(0)}) \otimes \phi(v_{(1)}) = \sum_{(v)} v_{(0)} \otimes (\pi \otimes id)\Delta_G(\phi(v_{(1)})),$$

which follows from straightforward calculation. Let us now define $I_\phi : V \otimes B_K \to \mathrm{ind}_K^G(\rho_\mathrm{R})$ as the linear extension of $v \otimes b \to T_\phi(v) \cdot b$. We want to prove that I_ϕ is bijective, which we will do by showing that its bilateral inverse is the linear extension of

$$I^\phi(v \otimes f) = \sum_{(f)} v \otimes \phi^{-1}(\pi(f_{(1)}))f_{(2)} = \sum_{(v)} v_{(0)} \otimes \phi^{-1}(v_{(1)})f.$$

On the one hand

$$I^\phi(I_\phi(v \otimes b)) = \sum_{(v)} I^\phi(v_{(0)} \otimes \phi(v_{(1)}))b$$
$$= \sum_{(v)(b)(\phi(v_{(1)}))} v_{(0)} \otimes \phi^{-1}(\pi(\phi(v_{(1)})_{(1)}b_{(1)}))\phi(v_{(1)})_{(2)}b_{(2)}$$
$$= \sum_{(v)(\phi(v_{(1)}))} v_{(0)} \otimes \phi^{-1}(\pi(\phi(v_{(1)}))_{(1)})\phi(v_{(1)})_{(2)}b$$
$$= \sum_{(v)} v_{(0)} \otimes \phi^{-1}(v_{(1)_{(1)}})\phi(v_{(1)_{(2)}})b = v \otimes b$$

The fact that I^ϕ takes its values in $V \otimes B_K$ follows from the first part of the proof. Finally,

$$I_\phi\left(\sum_{(f)} v \otimes \phi^{-1}(\pi(f_{(1)}))f_{(2)}\right) = I_\phi\left(\sum_{(v)} v_{(0)} \otimes \phi^{-1}(v_{(1)})f\right)$$
$$= \sum_{(v)} v_{(0)} \otimes \phi(v_{(1)_{(1)}})\phi^{-1}(v_{(1)_{(2)}})f = v \otimes f.$$

This proves the proposition. □

The second isomorphism of the proposition can be considered a morphism of $A_q(G)$-comodules provided we put on $V \otimes B_K$ the right coaction. Explicitly this is given by

$$\sigma_{\mathrm{R}} : V \otimes B_K \to V \otimes B_K \otimes A_q(G)$$
$$\sigma_{\mathrm{R}}(v \otimes b) = \sum_{(b)(\phi(v_{(2)}))(v)} v_{(0)}\phi^{-1}(v_{(0)_{(1)}})\phi(v_{(1)})_{(1)} \otimes b_{(1)} \otimes \phi(v_{(1)})_{(2)}b_{(2)}$$
$$= \sum_{(v)(b)} v_{(0)} \otimes b_{(1)} \otimes \phi(v_{(1)})b_{(2)}.$$

We also remark that in the case of *right sections* we would have obtained a B^K-module homomorphism for $\mathrm{ind}_K^G(\rho_{\mathrm{L}})$ both on the left and on the right.

Let us now consider the more delicate situation in which we start from a right corepresentation ρ_{R} of a left (resp. right) coisotropic quantum subgroup (C, π) of $A_q(G)$. Let B_C be the corresponding embeddable quantum homogeneous space. Then we can multiply functions on the space of induced representation by functions on the homogeneous space only on one side.

Lemma 3.5. *If $(A_q(K), \pi)$ is a left (resp. right) coisotropic quantum subgroup then $\mathrm{ind}_K^G(\rho_{\mathrm{R}})$ is a right (resp. left) sub-B_C-module (resp. sub B^C-module) of $V \otimes A_q(G)$.*

Proof. Let us consider the case of a right coisotropic subgroup. The claim follows from the following chain of equalities, for every $f \in A_q(G)$ and $b \in B_C$:

$$v \otimes (\pi \otimes id)\Delta(fb) = \sum_{(f)(b)} v \otimes \pi(f_{(1)}b_{(1)}) \otimes f_{(2)}b_{(2)}$$
$$= \sum_{(f)(b)} v \otimes \pi(f_{(1)}\varepsilon(b_{(2)})b_{(1)}) \otimes f_{(2)} = \sum_{(v)} v_{(0)} \otimes v_{(1)} \otimes fb.$$

This completes the proof. $\square$

Let us remark that the convolution product of linear maps $f, g : C \to A_q(G)$ is still well defined.

Definition 3.6. A right (resp. left) coisotropic subgroup (C, π) is said to have a section ϕ if there exists a linear map $\phi : C \to A_q(G)$ convolution invertible and such that:

(i) $\phi(\pi(1)) = 1$;

(ii) $\sum_{\phi(c)} \pi(\phi(c)_{(1)}u) \otimes \phi(c)_{(2)} = \sum_{(v)} \pi(v_{(1)}u) \otimes \phi(\pi(v_{(2)}))$ for every $c \in C$, $u \in A_q(G)$ and $v \in \pi^{-1}(c)$.

(resp. (ii′)) $\sum_{\phi(c)} \phi(c)_{(1)} \otimes \pi(u\phi(c)_{(2)}) = \sum_{(v)} \phi(\pi(v_{(1)}) \otimes \pi(uv_{(2)}))$ for every $c \in C$, $u \in A_q(G)$ and $v \in \pi^{-1}(c)$.

If, furthermore, ϕ is a right (resp. left) $A_q(G)$-module map then it is said to be a trivialization.

Lemma 3.7. *The convolution inverse of a right coisotropic subgroup section verifies*

$$\sum_{\phi(c)} \pi(\phi^{-1}(c)_{(1)}u) \otimes \phi^{-1}(c)_{(2)} = \sum_{(v)} \pi(S(v_{(2)}u)\phi^{-1}(\pi(v_{(1)}))),$$

where $c \in C$, $v \in \pi^{-1}(c)$.

Let us remark that $u = 1$ yields again condition (ii) of Definition 3.2.

Proposition 3.8. *Let us suppose that* (C, π) *is a right coisotropic subgroup with section* ϕ. *Then:*
(i) $A_q(G)$ *is isomorphic to* $C \otimes B_C$ *as a vector space;*
(ii) $\mathrm{ind}_K^G(\rho_R)$ *is isomorphic to* $V \otimes B_C$ *as a left* B_C*-module.*

Proof. Let $f \in A_q(G)$. Using Lemma 3.7 one can prove, like in Proposition 3.3, that $\sum_{(f)} \phi^{-1}(\pi(f_{(1)}))f_{(2)}$ belongs to B_C. The inverse isomorphisms A_ϕ and $A_{\phi^{-1}}$ are then realized exactly as in Proposition 3.7. Also the second part of the proof goes along exactly in the same way. □

Note that the B_C-module isomorphism of Propositions 3.3 and 3.8 proves that $\mathrm{ind}_K^G(\rho_R)$ is a projective B_C-module, which is a natural property to ask, as generalization of Swan's theorem, to spaces of sections of quantum bundles.

The explicit isomorphism of Proposition 3.8 can be used to describe the corepresentation directly on $V \otimes B_C$. The $\mathcal{F}_q(G)$-coaction τ_R on this space is given by

$$\begin{aligned}\tau_R(v \otimes b) &= \sum_{(b)(v)(v_{(1)})(\phi(v_{(2)}))} b_{(1)}\phi(v_{(2)})_{(1)} \otimes b_{(2)}\phi(v_{(2)})_{(2)}\phi^{-1}(v_{(1)}) \otimes v_{(0)} \\ &= \sum_{(v)(b)} b_{(1)} \otimes \phi(v_{(1)})b_{(2)} \otimes v_{(0)}.\end{aligned}$$

This allows a complete description of the induced corepresentation from the following data: the homogeneous space B_C and the section ϕ. This can be very useful in applications as the following examples will show.

Let us start with the non-standard Euclidean quantum group $E_\kappa(2)$ as described in [3]. The coisotropic subgroup we will consider is the coalgebra C with a denumerable family of group-like generators c_p, $p \in \mathbb{Z}$ and with restriction epimorphism

$$\pi(v^p) = c_p, \qquad \pi(a_1) = \pi(a_2) = 0$$

This is nothing but the quantum analogue of the circle subgroup. We remark that it cannot be given a quantum subgroup structure because such a π cannot be an algebra morphism [6]. The corresponding quantum homogeneous space is the κ-plane

$$[a_1, a_2] = \kappa(a_1 - a_2).$$

This embeddable quantum homogeneous space has an easily computed section

$$\phi : C \to E_\kappa(2), \qquad c_p \mapsto v^p.$$

Starting from one-dimensional irreducible corepresentations of this subgroup $\rho_n : 1 \to 1 \otimes c_n$ we obtain as corepresentation space the κ-plane with coaction

$$\mathrm{ind}_C^{E_\kappa(2)}(\rho_n)(a_1) = (v + v^{-1})v^n \otimes a_1 - \mathrm{i}(v - v^{-1})v^n \otimes a_2 + a_1 v^n \otimes 1,$$
$$\mathrm{ind}_C^{E_\kappa(2)}(\rho_n)(a_2) = \mathrm{i}(v - v^{-1})v^n \otimes a_1 + (v + v^{-1})v^n \otimes a_2 + a_2 v^n \otimes 1.$$

For $n = 0$ this is nothing but the regular corepresentation on the κ- plane. Using the duality pairing explicitly described in [3] it is easy to derive an algebra representation of the corresponding quantized enveloping algebra, which is never irreducible. The corresponding decomposition into irreducibles amounts to $E_\kappa(2)$-harmonic analysis on the κ-plane and has been carried through in [3] for the $n = 0$ case.

The second, similar, example is given by the standard Euclidean quantum group $E_q(2)$ with the family of right coisotropic subgroups (C, π_λ), where C is as before and the restriction epimorphism is given by

$$\pi(v^r n^s \bar{n}^k) = q^{2r(k+s)} \lambda^s \bar{\lambda}^k c_r (c_{-1}; q^{-2})_k (c_1; q^2)_s.$$

The corresponding quantum homogeneous spaces are called quantum hyperboloids and described in [1,9]; they are algebras in two generators, z and $\bar{z}$, with relation $z\bar{z} = q^2 \bar{z} z + (1 - q^2)$. For such coisotropic quantum subgroups there is a family of sections

$$\phi_r : C \to E_q(2), \qquad c_p \mapsto v^{p-r}.$$

Starting from one-dimensional irreducible corepresentations $\rho_m : 1 \mapsto 1 \otimes c_m$ we obtain $E_q(2)$-corepresentations on the quantum hyperboloids given by

$$\mathrm{ind}_C^{E_q(2)}(\rho_m)(z) = v^{m-r+1} \otimes z + v^m n \otimes 1,$$
$$\mathrm{ind}_C^{E_q(2)}(\rho_m)(\bar{z}) = v^{m+r-1} \otimes \bar{z} + v^m \bar{n} \otimes 1.$$

The decomposition into irreducibles for the infinite-dimensional corresponding $\mathcal{U}_q(e(2))$-representation is dealt with, in the $r = 0$-case, in [1].

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